NPS-MA-91-003

# NAVAL POSTGRADUATE SCHOOL

## Monterey, California

COVERING A CLOSED CURVE
WITH A GIVEN TOTAL CURVATURE

by

Mostafa Ghandehari

Technical Report for Period

April 1990-October 1990



Prepared for: Naval Postgraduate School
Monterey, CA 93943

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA 93943

Rear Admiral R. W. West, Jr.
Superintendent

Harrison Shull
Provost

This report was prepared in conjunction with research conducted for the Naval Postgraduate School and funded by the Naval Postgraduate School. 

Prepared by:

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

# REPORT DOCUMENTATION PAGE

Form Approved OMB No 0704 0188

| | |
|---|---|
| 1a REPORT SECURITY CLASSIFICATION: UNCLASSIFIED | 1b RESTRICTIVE MARKINGS |
| 2a. SECURITY CLASSIFICATION AUTHORITY | 3 DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited |
| 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE | |
| 4 PERFORMING ORGANIZATION REPORT NUMBER(S): NPS-MA-91-003 | 5 MONITORING ORGANIZATION REPORT NUMBER(S): NPS-MA-91-003 |
| 6a NAME OF PERFORMING ORGANIZATION: Naval Postgraduate School; 6b OFFICE SYMBOL (If applicable): MA | 7a NAME OF MONITORING ORGANIZATION: Naval Postgraduate School |
| 6c. ADDRESS (City, State, and ZIP Code): Monterey, CA 93943 | 7b ADDRESS (City, State, and ZIP Code): Monterey, CA 93943 |
| 8a. NAME OF FUNDING/SPONSORING ORGANIZATION: Naval Postgraduate School; 8b OFFICE SYMBOL (If applicable): MA | 9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: O&MN Direct Funding |
| 8c. ADDRESS (City, State, and ZIP Code): Monterey, CA 93943 | 10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO; PROJECT NO; TASK NO; WORK UNIT ACCESSION NO |

11 TITLE (Include Security Classification)
Covering a Closed Curve with a Given Total Curvature (U)

12 PERSONAL AUTHOR(S)
Mostafa Ghandehari

| 13a TYPE OF REPORT | 13b TIME COVERED | 14 DATE OF REPORT (Year, Month, Day) | 15 PAGE COUNT |
|---|---|---|---|
| Technical Report | FROM 4/90 TO 10/90 | 9 October 90 | 14 |

16 SUPPLEMENTARY NOTATION

| 17 COSATI CODES | | | 18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number) |
|---|---|---|---|
| FIELD | GROUP | SUB-GROUP | total curvature, closed curves, convexity |
| | | | |
| | | | |

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

It is known that if a closed curve C of class $C^2$ in $R^n$ is constrained to lie in a ball of radius R, then $L(C) \leq RK(C)$, where K(C) is the total curvature of C and L(C) denotes the Euclidean length of C. We show that

$$\left(\int_0^L |\kappa(s)|^p ds\right)^{1/p} \geq \frac{L^{1/p}}{R}.$$

where s is the arc length parameter and $\kappa$ denotes Euclidean curvature. Extensions of the above two inequalities to Minkowski spaces is discussed.

| | |
|---|---|
| 20 DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS | 21 ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED |
| 22a NAME OF RESPONSIBLE INDIVIDUAL: Mostafa Ghandehari | 22b TELEPHONE (Include Area Code): (408) 646-2124; 22c OFFICE SYMBOL: MA/Gh |

DD Form 1473, JUN 86 Previous editions are obsolete SECURITY CLASSIFICATION OF THIS PAGE

S/N 0102-LF-014-6603 UNCLASSIFIED

# COVERING A CLOSED CURVE WITH A GIVEN TOTAL CURVATURE

Mostafa Ghandehari

Department of Mathematics

Naval Postgraduate School

Monterey, California 93943

## ABSTRACT

It is known that if a closed curve $C$ of class $C^2$ in $R^n$ is constrained to lie in a ball of radius $R$, then $L(C) \leq RK(C)$, where $K(C)$ is the total curvature of $C$ and $L(C)$ denotes the Euclidean length of $C$. Assume $p \geq 1$. We show that

$$\left(\int_0^L |\kappa(s)|^p ds\right)^{1/p} \geq \frac{L^{1/p}}{R}.$$

where $s$ is the arc length parameter and $\kappa$ denotes Euclidean curvature. Extensions of the above two inequalities to Minkowski spaces is discussed.

## 1 INTRODUCTION

Let $C$ be a closed curve of class $C^2$ in Euclidean $n$-space $R^n$. We write the equations of $C$ as $x = x(s)$, $0 \leq s \leq L(C)$, where $s$ denotes Euclidean arc length and $L(C)$ is the Euclidean length of $C$. Denoting differentiation with respect to $s$ by a dot, we define the total curvature of $C$ as

$$K(C) = \int_C |\ddot{x}| ds = \int_0^L |\kappa(s)| ds.$$

Chakerian [4, 5] proves that if $C$ is constrained to lie in a ball of radius $R$, then

$$L(C) \leq RK(C). \tag{1}$$

The curves for which equality holds in (1) are circles of radius $R$ traversed a certain number of times.

In this article we prove the following theorem concerning the $p^{\text{th}}$ mean of curvature in Euclidean spaces and consider generalizations of (1) for a closed curve $C$ in a Minkowski plane (Minkowski spaces are simply finite dimensional normed linear spaces).

**Theorem 1.** Let $C$ be a close curve of class $C^2$ in Euclidean $n$ space $R^n$. Let $\kappa(s)$ and $K(C)$ denote the curvature and the total curvature of $C$ respectively. Let $L(C)$ denote the length of $C$. Assume $p \geq 1$, and that $C$ is constrained to be in a ball of radius $R$. Then

$$\left(\int_0^L |\kappa(s)|^p ds\right)^{1/p} \geq \frac{L^{1/p}}{R}, \tag{2}$$

Curves for which equality holds are circles of radius $R$ transversed a certain number of times.

Preliminary definitions and concepts are discussed in Section 2. The proof of Theorem 1 and related results are given in Section 3.

## 2 PRELIMINARIES

By a *convex body* in $R^n$ we mean a compact convex subset of $R^n$ with nonempty interior. For each direction $u \epsilon S^{n-1}$, where $S^{n-1}$ is the unit sphere centered at the origin in $R^n$, we let $h(K,u)$ denote the *support function* of the convex body $K$ evaluated at $u$. Thus, $h(K,u) = \sup\{u \cdot x : x \epsilon K\}$, which may be interpreted as the distance from the origin to the supporting hyperplane of $K$ having outward-pointing normal $u$. For a plane convex body we use the notation $h(K,\theta) = h(K,u)$, where $u = (\cos\theta, \sin\theta)$. The polar dual (or polar reciprocal) of a convex body $K$, denoted by $K^*$, is another convex body having the origin as an interior point and having the property that

$$h(K^*,u) = \frac{1}{r(K,u)} \quad \text{and} \quad r(K^*,u) = \frac{1}{h(K,u)}, \tag{3}$$

where $r(K,u)$ is the radial function of $K$ in the direction $u$.

Eggleston [8], pp. 25–28 contains definitions and some properties of polar duals.

Minkowski distance defined by means of a convex body was developed by Minkowski [11]. The articles by Busemann [1] and Petty [12] contain basic concepts for the study of Minkowskian geometry. We take a "unit circle" $E$ for the Minkowski plane to be a centrally symmetric convex body with its center at the origin in the Euclidean plane. The *Minkowski distance* from $x$ to $y$ is

$$||x-y|| = \frac{||x-y||_e}{r}, \tag{4}$$

where $||x-y||_e$ is the Euclidean length from $x$ to $y$, and $r$ is the Euclidean distance of the center of symmetry to the boundary of $E$ in the direction of the vector $y-x$.

Let $P$ be a polygonal path with succesive vertices $x_0, x_1, \ldots, x_n$. The Minkowskian length of $p$ with respect to the unit ball $E$ is defined by

$$\mu_E(P) = \sum_{i=1}^{n} ||x_i - x_{i-1}||, \tag{5}$$

where $||\cdot||$ is the Minkowskian norm.

Let $C$ be a rectifiable path. *The Minkowskian length of* $C$ with respect to the unit ball $E$ is defined by

$$\mu_E(C) = \sup_{P\epsilon\mathcal{P}} \mu_E(p) \tag{6}$$

where $p$ is the set of all polygonal paths inscribed in $C$.

Assume now that $C$ is continuously differentiable. If we let $ds_e(C,u)$, $u\epsilon S^{n-1}$ denote the Euclidean element of arc at a point where the tangent vector to the curve $C$ has direction $u$, then the Minkowskian element of arc, is denoted by

$$ds_m(C,u) = \frac{ds_e(C,u)}{r(E,u)}. \tag{7}$$

We can use (5), (6), and (7) to find the following expression for the Minkowskian length of $C$, denoted $\ell(C)$.

$$\ell(C) = \mu_E(C) = \int_C \frac{ds_e(C,u)}{r(E,u)} = \int_C ds_m(C,u). \tag{8}$$

Turning to the case of curves in $R^2$, we let $ds(C,u) = ds(C,\theta)$, where $u = (\cos\theta, \sin\theta)$. The *self-circumference* of the unit circle $E$ is the Minkowskian length of $E$ measured with respect to $E$ and is given by

$$\mu_E(\partial E) = \int_0^{2\pi} \frac{ds_e(E,\theta)}{r(E,\theta)}. \tag{9}$$

Gotab [10] was apparently the first to prove that

$$6 \leq \mu_E(\partial E) \leq 8. \tag{10}$$

Equality on the left occurs if and only if $E$ is an affine regular hexagon, and on the right if and only if $E$ is a parallelogram.

Schäffer [14], and independently later, Thompson [15], proved that

$$\mu_E(\partial E) = \mu_{E^*}(\partial E^*). \tag{11}$$

where $E^*$ is the polar dual of $E$.

A discussion of self-circumference of a plane convex body and its relationship to polar dual is given in Chakerian [2]. Some properties of the self-circumference for nonsymmetric Minkowski spaces is given in Chakerian and Talley [3].

Assuming that the boundary of the unit circle $E$ has nowhere zero Euclidean curvature, we define the *Minkowskian curvature* of a curve $C$ at a point $p$ by

$$\kappa_m(C,p) = \frac{\kappa_e(C,p)}{\kappa_e(E,\overline{p})} \tag{12}$$

where $\kappa_e(C,p)$ and $\kappa_e(E,\overline{p})$ denote the Euclidean curvatures of $C$ and $E$ at points $p$ and $\overline{p}$ respectively such that the unit tangent to $C$ at $p$ is parallel to the unit tangent to $E$ at $\overline{p}$.

[Other works define Minkowskian curvature differently. See for example Busemann [1]. The definition given here was used to prove the following Theorem 2 concerning curves with bounded Minkowskian curvature. A proof of Theorem 2 is contained in [9]. We shall use Theorem 2 in the next section.]

**Theorem 2.** Let $E$ be a unit circle in a Minkowski plane. Let $C$ be any continuously differentiable closed curve with length $\ell(C)$ (measured in the Minkowski metric). Assume $|\kappa_e(C,\cdot)| \leq k\kappa_e(E,\cdot)$ where $\kappa_e(C,\cdot)$ and $\kappa_e(E,\cdot)$ denote the Euclidean curvature. Then $C$ can be contained in a similar copy of the unit disk translated and magnified by a factor

$$\mu \geq \frac{\ell(C)}{4} - \frac{1}{4k}(\ell(E) - 4).$$

We use techniques from integral geometry. Santaló [13] is a good reference for integral geometry in Euclidean spaces. Given a curve $C$ in the Euclidean plane, let $L$ denote the length of $C$. Crofton's simplest formula, see Santaló [13], is

$$\int\int n dp d\theta = 2L, \tag{13}$$

where the integral is taken over all lines intersecting $C$. The pair $(p, \theta)$ is the polar coordinate representation of the foot of the perpendicular from the origin to the line, and $n$ is the number of intersections of a line with coordinates $(p, \theta)$ with $C$. The differential element $dG = dpd\theta$ is the *integral geometric density for the line.*

Chakerian [6] treats integral geometry in the Minkowski plane. We sketch the definitions he uses to develop Crofton's simplest formula in the Minkowski plane. Assume $E$ is "sufficiently" differentiable and has positive finite curvature everywhere. Parameterize $E$ by twice its sectorial area $\phi$ and write the equation of $E$ as

$$t = t(\phi), \qquad 0 \leq \phi \leq 2\pi, \qquad ||t|| = ||t - 0|| = 1.$$

$E$ is called the *indicatrix.* Define the *isoperimetrix* $T$ by the parametric representation

$$n(\phi) = \frac{dt(\phi)}{d\phi}, \qquad 0 \leq \phi \leq 2\pi.$$

Define $\lambda(\phi)$ by $\frac{dn(\phi)}{d\phi} = -\lambda^{-1}(\phi)t(\phi)$. Then the density for lines in two-dimensional Minkowski spaces is defined as follows. Let $G = G(p, \phi)$ be parallel to the direction $t(\phi)$. The equation of $G$ is

$$[t(\phi), x] = p,$$

where $[x, y] = x_1y_2 - x_2y_1$. Then the *density dG for lines* is

$$dG = \lambda^{-1}(\phi)dpd\phi.$$

It is then shown in Chakerian [6] that the simplest formula of Crofton holds:

$$\int ndG = 2\ell$$

where $n$ is the number of intersections of a line $G$ with a curve $C$, integration is taken over all lines intersecting $C$ and $\ell$ is the Minkowskian length of $C$.

# 3 RESULTS

In this section, we prove Theorem 1 concerning the $p^{\text{th}}$ mean of curvature in Euclidean spaces and consider generalization of (1) for a closed curve $C$ in a Minkowski plane.

**Proof of Theorem 1.** We can use Hölder's inequality to write

$$K(C) = \int_0^L 1|\kappa(s)|ds \leq \left(\int_0^L 1ds\right)^{1/q} \left(\int_0^L |\kappa(s)|^p ds\right)^{1/p} \tag{14}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Hence $\dfrac{1}{q} = 1 - \dfrac{1}{p} = \dfrac{p-1}{p}$. Multiplying both sides of (14) by $R$ and using (1) we obtain

$$L \leq R\,L^{p-1/p} \left(\int_0^L |K(s)|^p ds\right)^{1/p}, \tag{15}$$

giving the desired inequality (2). Equality holds if and only if equality holds in (14) and (1). For equality to hold in (14), $|\kappa(s)|^p$ has to be constant. The case of equality in (1) implies that equality holds in (2) if and only if $C$ is a circle of radius $R$ transversed a certain number of times. ■

We now give a probabilistic interpretation of (2) for the case $p = 2$. Recall that the variance of a random variable is non-negative, and that the variance is equal to the second moment minus the square of the mean. Consider the uniform probability distribution of $\frac{1}{L}$

on the closed curve of length $L$. Assume $|\kappa(s)|$ is a random variable. Denote the variance by $\sigma^2$. Then

$$\sigma^2 = \int_0^L \frac{1}{L}|\kappa(s)|^2 ds - \left(\int_0^L \frac{1}{L}|\kappa(s)| ds\right)^2 \geq 0.$$

Use the above inequality and (1) to obtain

$$\int_0^L |\kappa(s)|^2 ds \geq \frac{\left(\int_0^L |\kappa(s)| ds\right)^2}{L} \geq \frac{\left(\frac{L}{R}\right)^2}{L} = \frac{L}{R^2},$$

the desired result.

The following theorem is a generalization of (1) in a Minkowski plane. We emphasize that $K(C)$ is the total Euclidean curvature of a closed curve $C$. The proof given here is similar to Chakerian [4] for the inequality (1).

**Theorem 3.** Consider a $C^2$ closed curve $C$ in a Minkowski space with the unit ball $E$. Let $s$ and $s_m$ denote Euclidean and Minkowskian arc lengths respectively. Let $L(C)$ and $\ell(c)$ denote Euclidean and Minkowskian lengths of $C$. Let $x = x(s)$, $o \leq s \leq L(C)$ be a parametric representation of $C$. Let "dot" denote differentiation with respect to $s$. Then

$$\ell(C) \leq RK(C) + \int_C \left|\frac{\dot{r}}{r}\right| ds, \tag{16}$$

where $K(C)$ denotes the total Euclidean curvature of $C$, $r$ is the radius of $E$ in the direction of arc length $ds$, and $C$ is constrained to lie in a copy of $E$ magnified by a factor of $R$.

**Proof.** The Minkowskian arc length is given by $ds_m = \frac{ds}{r}$. Since $s$ is Euclidean arc length, $\dot{x}(s) \cdot \dot{x}(s) = 1$. Hence we can write $\ell(C)$ as follows

$$\ell(C) = \int_C ds_m - \int_C \frac{ds}{r} = \int_C \frac{1}{r}\dot{x} \cdot \dot{x} ds. \tag{17}$$

Using integration by parts and the triangle inequality, we obtain

$$\begin{aligned} \ell(C) &= x \cdot \frac{\dot{x}}{r}\Big|_0^{L(C)} - \int_C x \cdot \left(\frac{\dot{x}}{r}\right)^{\cdot} ds \\ &= -\int_C x \cdot \left(\frac{\dot{x}}{r}\right)^{\cdot} ds \leq \int ||x||_e \left|\left|\frac{\ddot{x}r - \dot{r}\dot{x}}{r^2}\right|\right|_e ds. \end{aligned} \tag{18}$$

The fact that $C$ lies in a copy of $E$ magnified by a factor of $R$ implies

$$||x||_e \leq Rr. \tag{19}$$

Using (18) and (19) we obtain

$$\begin{aligned}
\ell(C) &\leq \int_C ||x||_e \left\|\frac{\ddot{x}r - \dot{r}\dot{x}}{r^2}\right\|_e ds \\
&\leq \int_C Rr \left\|\frac{\ddot{x}r - \dot{r}\dot{s}}{r^2}\right\|_e ds \\
&\leq \int_C |\ddot{x}| ds + \int_C \left\|\frac{\dot{r}\dot{x}}{r}\right\|_e ds = RK(C) + \int_C \left|\frac{\dot{r}}{r}\right| ds. \quad \blacksquare
\end{aligned}$$

In Euclidean spaces the radius of the unit ball in each direction is constant and $\dot{r} = 0$. Hence (16) implies (1).

Fenchel's theorem states that the total curvature of a closed space curve $C$ is greater than, or equal to $2\pi$. It is equal to $2\pi$ if and only if $C$ is a plane convex curve. See Chern [7] for an elementary discussion of Fenchel's theorem. The following theorem shows that the total Minkowskian curvature of a closed convex curve is equal to the self-circumference of the unit circle $E$. Recall that

$$\kappa_m(C,\theta) = \frac{\kappa(C,\theta)}{\kappa(E,\theta)}.$$

**Theorem 4.** Let $C$ be a closed convex curve in a Minkowski plane with the unit circle $E$. Then the total Minkowskian curvature of $C$ is equal to the self-circumference of the unit ball $E$.

**Proof.** Let $\theta$ be the angle between the tangent to $C$ and the horizontal. Then,

$$\begin{aligned}
\int_C \frac{\kappa(C,\theta)}{\kappa(E,\theta)} ds_m(C,\theta) &= \int_C \frac{\kappa(C,\theta)}{\kappa(E,\theta)} \cdot \frac{1}{r(E,\theta)} ds(C,\theta) \\
&= \int_C \frac{d\theta}{\kappa(E,\theta) r(E,\theta)} = \int_C \frac{R(E,\theta)d\theta}{r(E,\theta)} \\
&= \int \frac{ds(E,\theta)}{r(E,\theta)} = L(E),
\end{aligned}$$

where we have used the fact that $ds = \kappa(C,\theta)ds(C,\theta)$ and that $R(E,\theta)d\theta = ds(E,\theta)$. $\blacksquare$

We relate this theorem to the inequality of Theorem 2 as follows: Let $\ell(\overline{C})$ be the Minkowskian length of $\overline{C}$, the convex hull of a closed curve $C$ with bounded Minkowskian curvature $|\kappa_m(C)| \leq k$. It is shown in [9] that $\overline{C}$ has the same bound for the Minkowskian curvature. Hence using Theorem 4 we obtain

$$\ell(E) = \int_0^{\ell(\overline{C})} \kappa_m(\overline{C}) ds_m \leq \int_0^{\ell(\overline{C})} k ds_m \leq k\ell(\overline{C}). \tag{20}$$

By a result in [9], $\ell(\overline{C}) \leq \ell(C)$. Hence

$$\ell(C) \geq \ell(\overline{C}) \geq \frac{1}{k}\ell(E) = \ell\left(\frac{1}{k}E\right). \tag{21}$$

By using Blaschke's Rolling Theorem, one can move a copy of $\frac{1}{k}E$ inside $\overline{C}$. The above inequality makes it plausible.

Given a convex curve $C$ in a Minkowski plane, one can easily generalize (1) and (2) for a Minkowski plane as follows. Suppose $RE$ covers a convex curve $C$. Then

$$\ell(C) \leq \ell(RE) = R\ell(E). \tag{22}$$

But since $\ell(E)$ is the total Minkowskian curvature, (22) is the Minkowskian analogue of (1). Using the same argument as in Theorem 1, we can use (22) and prove that for a convex closed curve in a Minkowski plane,

$$\left(\int_0^{\ell} |\kappa_m^p(C)| ds_m\right)^{1/p} \geq \frac{\ell(E)^{1/p}}{R}. \tag{23}$$

Theorem 5 gives the total Minkowskian curvature of a closed curve $C$ in terms of an integral involving the Euclidean arc length along the unit circle $E$ and the support function of the isoperimetrix $T$. Recall that the isoperimetrix was defined in page 5.

**Theorem 5.** Consider a closed curve $C$ in a Minkowski plane with unit circle $E$ and isoperimetrix $T$. Let $\nu(\theta)$ be the number of points where lines parallel to a fixed direction $\theta$ are tangent to the curve $C$. Then

$$\int |\kappa_m(C,\theta)| ds_m(C,\theta) = \int_C \nu(\theta) h(T,\theta) ds(E,\theta), \tag{24}$$

where $h(T,\cdot)$ is the support function of $T$, $ds(E,\cdot)$ is the Euclidean arc length of $E$ and $\kappa_m(C,\cdot)$ is the Minkowskian curvature.

**Proof.** Using the definition of Minkowskian arc length and the fact that $ds(C,\theta)$ is parallel to the direction $\theta + \frac{\pi}{2}$ we obtain

$$ds_m(C,\theta) = \frac{ds(C,\theta)}{r\left(E, \theta + \frac{\pi}{2}\right)}. \tag{25}$$

The isoperimetrix is the same as the polar dual rotated 90 degrees. Hence we can use (25) and (3) to obtain

$$ds_m(C,\theta) = \frac{ds(C,\theta)}{r\left(E, \theta + \frac{\pi}{2}\right)} = h(T,\theta)ds(C,\theta). \tag{26}$$

Using the definition of Minkowskian curvature we obtain

$$\kappa_m(C,\theta) = \frac{\kappa_e(C,\theta)}{\kappa_e(E,\theta)} = \frac{\frac{d\theta}{ds(C,\theta)}}{\frac{d\theta}{ds(E,\theta)}} = \frac{ds(E,\theta)}{ds(C,\theta)}. \tag{27}$$

Let $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a partition of $C$ such that the tangent map is monotonic on each $\sigma_i$ (see Figure 1). Using (26), (27), and the partition $\{\sigma_i\}_{i=1}^n$, we can calculate the total Minkowskian curvature as follows:

$$\int |\kappa_m(C,\theta)| ds_m(C,\theta) = \int_C \left| \frac{ds(E,\theta)}{ds(C,\theta)} \right| h(T,\theta) ds(C,\theta)$$

$$= \int_C h(T,\theta) \left| \frac{ds(E,\theta)}{ds(C,\theta)} \right| ds(C,\theta) = \sum_{i=1}^{n} \int_{\sigma_i} h(T,\theta) ds(E,\theta).$$

The last expression in the net covering of the boundary of $E$ by the tangent map. We look at each point of the boundary of $E$ and see how many times it has been covered and integrate $h(T,\theta)$ over the boundary of $E$ counting this multiplicity. Thus the proof of (24) is completed. ∎

The following theorem is another generalization of (1) in a Minkowski plane. Recall that Theorem 3 was also a generalization.

**Theorem 6.** Consider a $C^2$ closed curve $C$ in a Minkowski plane with the unit circle $E$. Let $K_T(C)$ be the total Minkowskian curvature of $C$ measured with respect to isoperimetrix $T$. Then

$$\ell(C) \leq RK_T(C) \tag{28}$$

where $\ell(C)$ is the Minkowskian length of $C$ measured with respect to unit circle $E$ and $C$ is constrained to lie in a copy of $E$ magnified by a factor of $R$.

**Proof.** Let $G$ be a line parallel to the directions $\theta$. Recall that in Section 2 we parameterized $E$ by twice its sectorial area $\phi$ as

$$t = t(\phi), \qquad 0 \leq \phi \leq 2\pi, \qquad ||t|| = 1.$$

If we let $p = [t(\phi), x]$ where $[x, y] = x_1y_2 - x_2y_1$, then the density $dG$ for the line is $dG = \lambda^{-1}(\phi)dpd\phi = dpds_m = dp\frac{ds}{r}$ where $ds_m$ and $ds$ are Minkowskian and Euclidean arc lengths of the isoperimetrix. See Chakerian [6]. He uses $d\sigma$ and $d\sigma_e$ instead of $ds_m$ and $ds$.

We note that $P$ represents the area of the parallelogram formed by $t$ and $x$ for any $x$ on $G$. We can write this area as $pr$ where $p$ is the Euclidean distance from the origin to $G$ and $r = ||t||_e$. Thus we have

$$P = pr. \tag{29}$$

Using (29) we have

$$dP = rdp + pdr. \tag{30}$$

One can show that $drds_m = 0$. Hence

$$dG = dpds_m = rdpds_m = dpds, \tag{31}$$

so that we can write the Minkowskian length of $C$ as

$$\ell(C) = \frac{1}{2}\int n(p,\theta)dpds(T,\theta) \tag{32}$$

where $n(p,\theta)$ is the number of intersections of the line $G$ with $C$. Observing the fact that $n(p,\theta) \leq 2\nu(\theta)$ we have

$$\begin{aligned}
\ell(C) &\leq \frac{1}{2}\int 2\nu(p,\theta)dpds(T,\theta) \\
&\leq \frac{1}{2}\cdot 2\int \nu(p,\theta)h(RE,\theta)ds(T,\theta) \\
&= R\int \nu(p,\theta)h(E,\theta)ds(T,\theta).
\end{aligned}$$

The last integral is the total curvature of $C$ measured with respect to the isoperimetrix. Thus the proof is completed. ∎

## REFERENCES


1. Busemann, H. The foundations of Minkowskian geometry, *Comm. Math. Helv.*, **24**, (1950), pp. 156–187.

2. Chakerian, G. D. Mixed areas and the self-circumference of a plane convex body, *Arch. Math.*, **34**, (1980), pp. 81–83.

3. Chakerian, G. D. and Talley, W. K. Some properties of the self-circumference of convex sets, *Arch. Math.*, **20**, (1969), pp. 431–443.

4. Chakerian, G. D. On some geometric inequalities, *Proc. Amer. Math. Soc.*, **15**, (1964), pp. 886–888.

5. Chakerian, G. D. An inequality for closed space curves, *Pacific J. Math.*, **12**, (1962), pp. 53–57.

6. Chakerian, G. D. Integral geometry in the Minkowski plane, *Duke Math. J.*, **29**, (1962), pp. 375–382.

7. Chern, S. S. Curves and surfaces in Euclidean space, *Studies in Global Geometry and Analysis, Studies in Math.*, vol. 4, MAA, 1967.

8. Eggleston, H. G. *Convexity*, Cambridge Univ. Press, Cambridge, 1958, pp. 25–28.

9. Ghandehari, M. Plane curves with restricted curvature in the Minkowski plane, Technical report, Naval Postgraduate School, NPS-53-90-010, September 1990.

10. Gotab, S. Quelque problèmes métrique de la géométrie de Minkowski, *Travaux de l'Académie des Mins a' Cracovie*, **6**, (1932), (Polish, French summary).

11. Minkowski, H. Theorie der Konvexen Körper, Insbesondre Begründung Ihres oberflächen Bgriffs, *Ges Abhandl*, Leipzig-Berlin, vol. 2, (1911), pp. 131–229.

12. Petty, C. M. On the geometry of the Minkowski plane, *Riv. di Mat.*, parma 6, (1955), pp. 269–292.

13. Santaló, S. L. A. *Introduction to Integral Geometry*, Paris, Hermann, 1953.

14. Schäffer, J. J. The self-circumference of polar convex disks, *Arch. Math.*, **24**, (1973), pp. 87–90.

15. Thompson, A. C. An equiperimetric property of Minkowski circles, *Bull. London Math. Soc.*, **7**, (1975), pp. 271–272.

## ACKNOWLEDGMENT

This article was prepared for and funded by the Naval Postgraduate School Research Council.

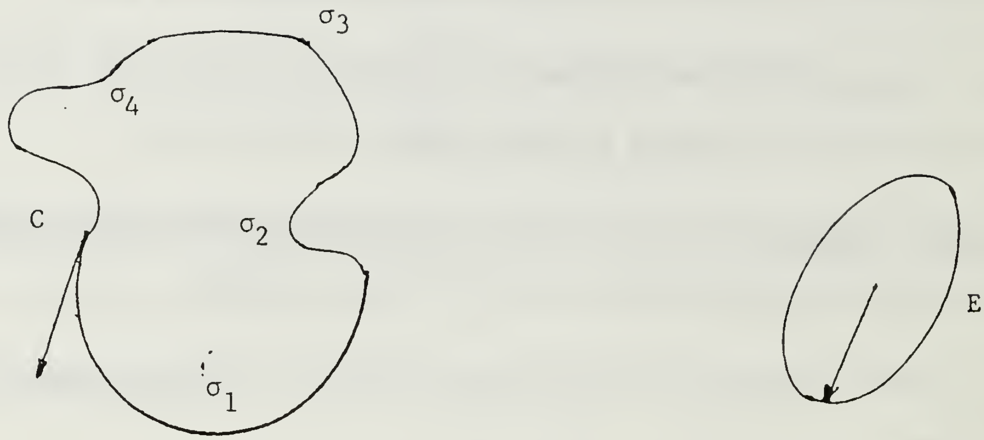


**Figure 1** The tangent map is monotonic on each $\sigma_i$.

## INITIAL DISTRIBUTION LIST

Professor Donald Albers
Department of Mathematics
Menlo College
1000 El Camino Real
Atherton, CA 94025

Professor G. L. Alexanderson
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053

Professor Gulbank Chakerian
Department of Mathematics
University of California
Davis, CA 95616

Professor Harold Fredricksen
Department of Mathematics
Naval Postgraduate School
Monterey, CA 93943

Prof. Mostafa Ghandehari (30)
Department of Mathematics
Naval Postgraduate School
Monterey, CA 93943

Professor Helmut Groemer
Department of Mathematics
University of Arizona
Tucson, AZ 85721

Professor David Logothetti
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053

Professor Erwin Lutwak (3)
Polytechnic Institute of
333 Jay Street
Brooklyn, NY 11201

Library, Code 0142 (2)
Naval Postgraduate School
Monterey, CA 93943

Professor Edward O'Neill
Department of Mathematics
and Computer Science
Fairfield University
Fairfield, CT 06430

Professor Jean Pedersen
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053

Professor Richard Pfiefer
Department of Mathematics
and Computer Science
San Jose State College
San Jose, CA 95192

Professor Thomas Sallee
Department of Mathematics
University of California
Davis, CA 95616

Professor Benjamin Wells
Department of Mathematics
Univ. of San Francisco
San Francisco, CA 94117

Professor James Wolfe
Department of Mathematics
University of Utah
Salt Lake City, UT 84112

Defense Technical Inf. (2)
Center
Cameron Station
Alexandria, VA 22214

Department of Mathematics
Code MA
Naval Postgraduate School
Monterey, CA 93943

Research Administration
Code 81
Naval Postgraduate School
Monterey, CA 93943